McGill University

Math 263-2011 Winter: Differential Equations for Engineers

Solutions to Final Exam

1.(**8 points**) Find the general solution for of the equation:

$$(x^2 + 3xy + y^2) - x^2 y' = 0.$$

Solution: This is a homogeneous equation, one has the general solution:

$$\frac{x}{x+y} + \ln|x| = C; \quad \text{and} \quad y = -x.$$

2.(**8 points**)Given the equation

$$(\mathrm{e}^x \sin y/y - 2\sin x)\mathrm{d}x + \frac{(\mathrm{e}^x \cos y + 2\cos x)}{y}\mathrm{d}y = 0.$$

1. Check whether or not it is exact EQ.
2. If it is not exact, find an integrating factor.
3. Derive the general solution.

Solution:

$$\mu(y) = y; \quad y(x) = \mathrm{e}^x \sin y + 2y\cos x = C.$$

3. (**12 points**) By using the method of differential operators, solve

$$y'' + 2y' + 2y = 2\mathrm{e}^{-x}\sin x.$$

1. Determine what is the annihilator of the inhomogeneous term;
2. Find a particular solution;
3. Write the general solution for the equation.

Solution:

$$\begin{aligned} Q(D) &= (D+1)^2 + 1\,, \\ y_p &= -xe^{-x}\cos x\,, \\ y &= e^{-x}(c_1\cos x + c_2\sin x - x\cos x) \end{aligned}$$

4.(**12 points**) Find the general solution for the EQ.

$$(x-1)y'' - xy' + y = \sin x, \quad (x > 1),$$

Given that $y_1(x) = \mathrm{e}^x$ satisfies the associated homogeneous Eq.

Solution: Reduction of order gives $y = x$ as another solution, so the general solution to the homogeneous equation is

$$y = c_1 x + c_2 e^x$$

Variation of parameters then gives $y_p = ux + ve^x$ where

$$u' = \frac{\sin x}{1-x}, \qquad v' = \frac{x \sin x}{e^x(x-1)}$$

and so the general solution is

$$y = c_1 x + c_2 e^x + x \int \frac{\sin x}{1-x} dx + e^x \int \frac{x \sin x}{e^x(x-1)} dx$$

5.(**10 points**) Find the Laplace transform $\mathcal{L}(f)$, for the function:

$$f(t) = \begin{cases} 0, & t < 1; \\ (t-1)^2, & 1 \le t < 2, \\ (3-t), & 2 \le t < 3, \\ \mathrm{e}^{-t}, & t \ge 3. \end{cases}$$

Solution: Writing $f(t)$ as

$$\begin{aligned} f(t) &= (t-1)^2 u_1(t) - ((t-1)^2 - (3-t))u_2(t) + (e^{-t} + t - 3)u_3(t) \\ &= (t-1)^2 u_1(t) - ((t-2)^2 + 3(t-2))u_2(t) + (e^{-3}e^{-(t-3)} + t - 3)u_3(t) \end{aligned}$$

gives

$$\begin{aligned} \mathscr{L}\{f(t)\} &= e^{-s}\mathscr{L}\{t^2\} - e^{-2s}\mathscr{L}\{t^2 + 3t\} + e^{-3s}\mathscr{L}\{e^{-3}e^{-t} + t\} \\ &= e^{-s}\frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right) + e^{-3s}\left(\frac{e^{-3}}{s+1} + \frac{1}{s^2}\right) \end{aligned}$$

6.(**10 points**) Find the inverse Laplace transform for the function:

$$F(s) = \frac{(s-2)\mathrm{e}^{-s}}{s^2 - 4s + 3}.$$

Solution:

$$f(t) = \frac{1}{2}\big[e^{3(t-1)} + e^{t-1}\big]u_1(t).$$

7.(**10 points**) Solve the IVP:

$$y'' + 4y = \sin t + u_\pi(t)\sin(t-\pi); \quad y(0) = 0,\, y'(0) = 0.$$

Solution:

$$y(t) = \frac{1}{6}(2\sin t - \sin 2t) - \frac{1}{6}u_\pi(t)(2\sin t + \sin 2t)$$

8.(**15 points**) Given the Eq:

$$2(1+x)y'' + \alpha y' - (x+\sin x)y = 0.$$

1. Find its regular singular points x_0; determine the exponent at such singular point $x = x_0$.
2. Determine the **forms** of two linear independent solutions near the regular singular point, corresponding to different values of α.
3. Demonstrate the qualitative behavior of the solutions near the singular point for the cases of $\alpha = 0, \pm 1$, indicate:
   - are all solutions bounded?
   - are all solutions are unbounded?
   - are some solutions bounded?

Solution: Regular singular point: $x = -1$. To determine if it is regular, compute:

$$\lim_{x\to -1}(x+1)p(x) = \lim_{x\to -1}\frac{(x+1)\alpha}{2(1+x)} = \frac{\alpha}{2}\,.$$

Similarly,

$$\lim_{x\to -1}(x+1)^2 q(x) = \lim_{x\to -1}\frac{(x+1)^2(x+\sin x)}{2(x+1)} = 0\,.$$

Thus the singular point is indeed regular, and

$$p_0 = \frac{\alpha}{2}\,, \qquad q_0 = 0\,.$$

Thus the indicial equation, for general n, is

$$r\left(r - 1 + \frac{\alpha}{2}\right) = 0\,,$$

which gives roots of

$$r_1 = 1 - \frac{\alpha}{2}, \qquad r_2 = 0.$$

Now if α is not an even integer, these two roots are distinct and do not differ by an integer, two linearly independent solutions are

$$y_1 = \sum_{n=0}^{\infty} a_n(0)(x+1)^n$$

$$y_2 = |x+1|^{r_1} \sum_{n=0}^{\infty} a_n(r_1)(x+1)^n.$$

If $\alpha = 2$, two linearly independent solutions are

$$y_1 = \sum_{n=0}^{\infty} a_n(0)(x+1)^n$$

$$y_2 = y_1(x)\ln|x+1| + \sum_{n=1}^{\infty} a_n'(0)(x+1)^n.$$

If $\alpha \neq 2$ but α is an even integer, two linearly independent solutions are

$$y_1 = \sum_{n=0}^{\infty} a_n(0)(x+1)^n$$

$$y_2 = a y_1(x)\ln|x+1| + \sum_{n=0}^{\infty} b_n(x+1)^n.$$

Now for $\alpha = 0$, we are in the last case, and it is seen that $y_1(x) \sim a_0$, which is bounded, but y_2 is unbounded, as $x \to -1$. For $\alpha = 1$, we are in the first case, and it is seen that $y_1(x) \sim a_0$, which is bounded, and $y_2 \sim |x+1|^{1/2} \sim 0$, so both solutions are bounded as $x \to -1$. For $\alpha = -1$, we are again in the first case, and it is seen that $y_1(x) \sim a_0$, which is bounded, and $y_2 \sim |x+1|^{3/2} \sim 0$, so both solutions are bounded as $x \to -1$.

9.(**10 points**) Given the following IVP of the system of Eq's:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A\mathbf{x}, \quad \mathbf{x}(0) = (7,5,5)^T,$$

where

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{bmatrix}.$$

- Use the method of eigenvalues and eigenvectors of matrix to solve the problem.

- There is another general method to solve this problem. Demonstrate what that method is?

Solution:

$$\mathbf{x}(t) = 6\mathrm{e}^{t}(1, 2, -1)^{T} + 3\mathrm{e}^{-t}(1, -2, 1)^{T} + \mathrm{e}^{4t}(-2, -1, 8)^{T}.$$